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Distribution of complex phase velocities for small disturbances to pipe Poiseuille flow

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It is numerically known that normal modes for small disturbances to pipe Poiseuille flow have complex phase velocities c which form a Y-shaped set of discrete points in the fourth quadrant, when the Reynolds number R is large. In this paper, the eigenvalue problem of determining c for axisymmetric torsional disturbances is treated, and the Y-shaped distribution of these c is studied analytically (with a little aid from numerics) by using some asymptotic forms of a Whittaker function. As a result, a Y-shaped contour on which eigenvalues c are approximately located is obtained, independent of the wavenumber α and R, from simple equations which contain elementary functions only. Naturally, the location of each individual c depends on α and R. How it changes on the contour when large R is becoming still larger is explained. Several approximate values of c on the contour are compared with c computed by Schmid & Henningson (1994), and their agreement is seen to be good. Furthermore, the limit $\mathrm{Re}\, c \to 2/3$ as $\mathrm{Im}\, c \to -\infty$ is rigorously proved when R is fixed at an arbitrary number, which is not required to be large. This limit is shown to be true also with eigenvalues c for axisymmetric meridional disturbances. The alternate distribution of c for torsional and meridional disturbances on a branch of the Y-shaped contour is explained.

1. Introduction

The stability of Poiseuille flow in a circular pipe is a problem with a long history. The theoretical study of its dependence on the Reynolds number R was started by Sexl (1927). After that, many researchers investigated the behaviour of small disturbances to pipe flow from various theoretical viewpoints and deduced the linear stability at every value of R (see Drazin & Reid 1981 and the references therein). Consequently, researchers' interests moved into nonlinear effects of non-small disturbances. However, the linear stability problem has not been completely solved yet. In fact, since 1990s, the importance of careful studies on linear effects of small disturbances has been realized again in connection with pseudospectra of the pipe-flow version of the Orr–Sommerfeld operator (see Trefethen, Trefethen & Schmid 1999; Schmid & Henningson 2001 and the references therein).

Let us consider small axisymmetric disturbances for which normal modes have the factor $\mathrm{e}^{\mathrm{i}\alpha(x-ct)}$ ($\alpha > 0$, $c \in \mathbb{C}$) at time t in the cylindrical coordinate system (r, θ, x) with the x-axis parallel to the pipe Poiseuille flow ($0 < r < 1$). The wavenumber α is arbitrarily fixed throughout this paper. It is known that the complex phase velocities c, for which c_r ($:= \mathrm{Re}\, c) \in (0, 1)$ and c_i ($:= \mathrm{Im}\, c) < 0$, are classified into three families when R is large. The first family consists of c such that $c \to 1$ as $R \to \infty$. It was found

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by Pekeris (1948), and its corresponding normal modes are called 'centre' or 'fast' or 'P' modes. The second family consists of c such that $c \to 0$ as $R \to \infty$. It was found by Corcos & Sellars (1959), and its corresponding normal modes are called 'wall' or 'slow' or 'A' modes. Corcos & Sellars (1959) analytically studied these two families in detail in the case of c close to 0 or 1. The third family consists of c such that $c_r \approx 2/3$, and each c joins either the first or the second family when R becomes still larger. It was observed in numerical calculations of Davey & Drazin (1969), Salwen & Grosch (1972) and O'Sullivan & Breuer (1994), and its corresponding normal modes are called 'mean' or 'S' modes. These three families form a Y-shaped set of discrete points on the c-plane (Davey & Drazin 1969; Schmid & Henningson 1994 (table 1, $n = 0$), 2001 (p. 506, $n = 0$)). Interestingly, such Y-shaped structure is retained even if the axisymmetry of disturbances is broken (O'Sullivan & Breuer 1994; Schmid & Henningson 1994, 2001; Meseguer & Trefethen 2003).

As Schmid & Henningson (1994, 2001) and Meseguer & Trefethen (2003) pointed out, numerical calculations of linear effects of small disturbances are error-prone. Therefore, an analytical check on them is worth having, even if in a restricted case. Nevertheless, as far as is known, it was only Reid & Ng (2003) who made analytical investigation of the above three families in the case of c not close to 0 or 1.

The main aim of this paper is to deal with the eigenvalue problem of determining c for small torsional disturbances with axisymmetry and investigate analytically (with a little aid from numerics) the distribution of c not necessarily close to 0 or 1. For this, we make use of asymptotic forms of a Whittaker function (a confluent hypergeometric function of Whittaker's) which were derived by Skovgaard (1966). Our main results are as follows:

1. A Y-shaped contour on which discrete eigenvalues c are approximately located is obtained by numerical solution of simple equations which contain elementary functions only and are independent of α and R.
2. The limit $c_r \to 2/3$ as $c_i \to -\infty$ is rigorously proved when R is fixed at an arbitrary number, which is not required to be large.
3. It is interpreted how each c nearly on the Y-shaped contour moves towards the real axis when large R becomes still larger.
4. Several approximate values of c on the Y-shaped contour are compared with c obtained in the numerical calculation of Schmid & Henningson (1994, table 1, $n = 0$), and their agreement is seen to be good.

In addition, axisymmetric meridional disturbances are also discussed restricted to the mean-mode family:

5. The above-mentioned limit, $c_r \to 2/3$ as $c_i \to -\infty$, is shown to be also true with eigenvalues c for meridional disturbances.
6. It is analytically explained that c for torsional and meridional disturbances are alternately located on the mean-mode branch of the Y-shaped contour.

The application of a confluent hypergeometric function of Kummer or Whittaker to the stability analysis of pipe Poiseuille flow was done by Pekeris (1948) and Sexl & Spielberg (1958). In their times, however, the results of Skovgaard (1966) were not available. Reid & Ng (2003) dealt with not only torsional but also meridional disturbances by their own asymptotic analysis, part of which was essentially the same as of Skovgaard (1966). Although our disturbances are mostly torsional, we study the distribution of c more specifically than Reid & Ng (2003).

In §2, the formulation of our problem is given, and the behaviour of torsional disturbances is described by the Whittaker function. In §3, the region of c is broken into three sets and asymptotic forms of the Whittaker function are derived for each set by using a Bessel or an Airy function. In § 4, it is shown that the locations of zeros of the Bessel and the Airy functions account for the Y-shaped structure of c. The distribution of c for mean modes is discussed, and the limit $c \to 2/3 - \mathrm{i}\infty$ is proved in § 5. The distribution of c for wall and centre modes is treated in § 6. The limit $c \to 2/3 - \mathrm{i}\infty$ for meridional disturbances and the alternate distribution of c for torsional and meridional disturbances are shown in § 7. Some remarks are presented in § 8.

2. Formulation

Let G be a function of r, such that $G\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)}/r$ is a normal mode for axisymmetric torsional disturbances to the pipe flow which is parallel to the x-axis and has the velocity $1-r^2$ $(0<r<1)$. Pekeris (1948) derived the linearized equation

$$\frac{\mathrm{d}^2G}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}G}{\mathrm{d}r} - \alpha^2 G - \mathrm{i}\alpha R(1-r^2-c)G = 0. \tag{2.1}$$

As was mentioned in § 1, α (>0) is arbitrarily fixed. The boundary conditions are

$$G(1) = 0, \qquad \left|\lim_{r\to+0}\frac{G(r)}{r}\right| < \infty. \tag{2.2a, b}$$

We define the function μ by

$$\mu(R, c; r) = M_{\kappa,1/2}\left(\sqrt{\alpha R}\,\mathrm{e}^{-\mathrm{i}\pi/4}r^2\right) \quad \text{with} \quad \kappa = \frac{1}{4}\left[\frac{\sqrt{\alpha R}\,(1-c)}{\mathrm{e}^{\mathrm{i}\pi/4}} - \frac{\alpha^2\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{\alpha R}}\right]. \tag{2.3}$$

Here M is the first-kind Whittaker function, which is expressed with Kummer's confluent hypergeometric function ${}_1F_1$ as $M_{\kappa,1/2}(s) = s\,\mathrm{e}^{-s/2}{}_1F_1(1-\kappa; 2; s)$ (see Buchholz 1969, § 2, for basic properties of Whittaker functions). Then the solution to (2.1) with (2.2b) (or more restrictively, $\lim_{r\to+0} G/r = 0$) is written as $G = A\,\mu(R, c; r)$ with an arbitrary constant A. The eigenvalue c is determined by (2.2a); that is

$$\mu(R, c; 1) = 0. \tag{2.4}$$

The quotient of the first term of κ in (2.3) divided by the second has the absolute value $R|1-c|/\alpha$. If $R|1-c| \to \infty$, then κ is asymptotically equal to the k defined by

$$k = \frac{\sqrt{\alpha R}\,(1-c)}{4\mathrm{e}^{\mathrm{i}\pi/4}} = \frac{\sqrt{\alpha R}}{4|z|}\,\mathrm{e}^{-\mathrm{i}(\arg z+\pi/4)}.$$

Here $z = 1/(1-c)$, and from now on, this z will be frequently used for convenience. In the limit

$$\sqrt{R}\,|1-c| \to \infty \quad \text{and} \quad R|1-c| \to \infty, \tag{2.5}$$

we have $|k| \to \infty$ and $\mu(R, c; 1) \sim M_{k,1/2}(4kz)$, to which asymptotic results of Skovgaard (1966) are applicable.

3. Asymptotic forms of $\mu(R, c; 1)$ in the limit (2.5)

As Pekeris (1948) proved, every eigenvalue c determined by (2.4) satisfies $0 < c_r < 1$ and $c_i < 0$, in other words, $0 < \arg(1-c) < \pi/2$ and $0 < |1-c| < \sec\arg(1-c)$. Therefore, we restrict our attention to this region of c. Consequently, z belongs to

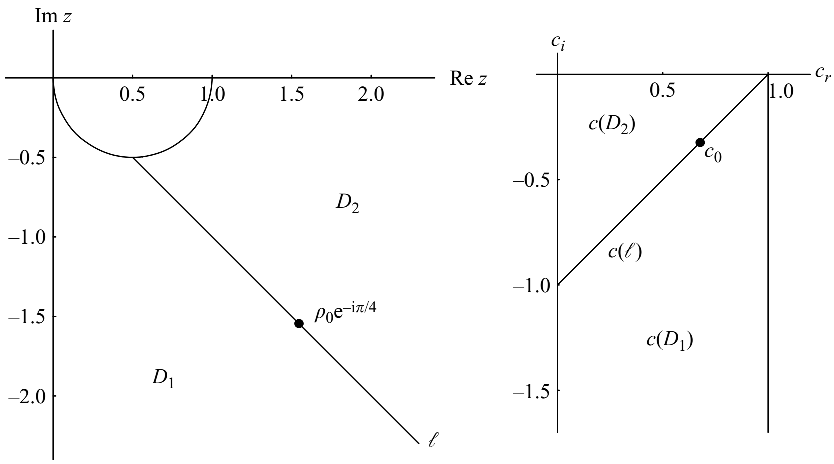

FIGURE 1. The sets D_1, D_2 and ℓ of z and the corresponding sets of $c = 1 - 1/z$. See (3.1) for the definition of ρ_0. The point c_0 corresponds to $z = \rho_0 e^{-i\pi/4}$; that is $c_0 = 1 - \rho_0^{-1} e^{i\pi/4}$.

one of the following three sets:

$$\begin{aligned} D_1 &= \left\{ s : -\pi/2 < \arg s < -\pi/4,\ \left|s - \tfrac{1}{2}\right| > \tfrac{1}{2},\ |s| < \infty \right\}, \\ D_2 &= \left\{ s : -\pi/4 < \arg s < 0,\ \left|s - \tfrac{1}{2}\right| > \tfrac{1}{2},\ |s| < \infty \right\}, \\ \ell &= \left\{ s : \arg s = -\pi/4,\ \left|s - \tfrac{1}{2}\right| > \tfrac{1}{2},\ |s| < \infty \right\}. \end{aligned}$$

Figure 1 shows the locations of D_1, D_2 and ℓ on the z-plane and the corresponding sets on the c-plane.

For the derivation of asymptotic forms of $\mu(R, c; 1)$ in the limit (2.5), it is convenient to define ξ by

$$\xi(z) = \begin{cases} \frac{1}{2} z^{1/2}(z-1)^{1/2} - \frac{1}{2} \ln\left[z^{1/2} + (z-1)^{1/2}\right] - i\pi/4 & \text{for } z \in D_1, \\ \frac{1}{2} z^{1/2}(z-1)^{1/2} - \frac{1}{2} \ln\left[z^{1/2} + (z-1)^{1/2}\right] & \text{for } z \in D_2 \cup \ell. \end{cases}$$

Here, and from now on, multi-valued functions should be understood to take their principal values. Figure 2(a, b) shows $\xi(D_1)$ and $\xi(D_2 \cup \ell)$ on the ξ-plane. Let us remark on the intersection of the line $\xi(\ell)$ and the negative imaginary axis in figure 2(b). It is at $\xi = \xi_0 := \xi(\rho_0 e^{-i\pi/4})$, where ρ_0 ($> 1/\sqrt{2}$) is a solution of

$$\mathrm{Re}\left\{ \left(\rho_0 e^{-i\pi/4}\right)^{1/2} \left(\rho_0 e^{-i\pi/4} - 1\right)^{1/2} - \ln\left[\left(\rho_0 e^{-i\pi/4}\right)^{1/2} + \left(\rho_0 e^{-i\pi/4} - 1\right)^{1/2} \right] \right\} = 0. \quad (3.1)$$

We numerically get

$$\rho_0 \approx 2.1844, \qquad \xi_0 \approx -0.55204\, i.$$

The intersection of the boundary of $\xi(D_1)$ and the negative imaginary axis in figure 2(a) is at $\xi = \xi_0 - i\pi/4$.

We now express $\mu(R, c; 1)$ asymptotically as follows:

1. Case: $z \in D_1$ ($0 < \arg k < \pi/4$). Noting that $z^{1/2} = (-z)^{1/2} e^{-i\pi/2}$ and $(z-1)^{1/2} = (1-z)^{1/2} e^{-i\pi/2}$ hold, ξ is thus rewritten as

$$\xi = \tfrac{1}{2} z^{1/2}(z-1)^{1/2} - \tfrac{1}{2} \ln\left[(-z)^{1/2} + (1-z)^{1/2}\right] \quad (3.2)$$

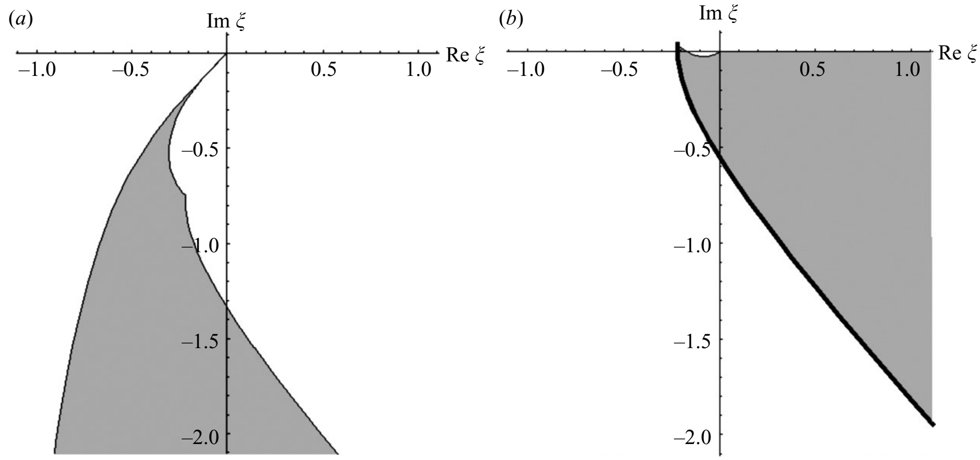

FIGURE 2. The range of ξ (grey-coloured regions). (*a*) $\xi(D_1)$; (*b*) $\xi(D_2)$ with the thick line $\xi(\ell)$.

(Skovgaard 1966; (4.7) in his work used this form; see also (4.11) in his work with $\mathrm{i}\pi/4$ replaced by $-\mathrm{i}\pi/4$); we have

$$\mu(R, c; 1) \sim 2(-\xi)^{1/2}\left(\frac{z}{z-1}\right)^{1/4} I_1(4k\xi) \tag{3.3}$$

in the limit (2.5). Here I_1 denotes the first-kind modified Bessel function of the first order. Skovgaard (1966, p. 75, line 6) left the factor $\mathrm{e}^{\mathrm{i}[\pi+\arg(z-1)]/4}$ in his $c_1(v, m)$, which should have been independent of z, and thus his asymptotic form (4.29) of $M_{v/4,m}(vz)$ contained $|z-1|^{-1/4}$ instead of $(z-1)^{-1/4}$. With the correction of this, we get (3.3), which has been verified numerically.

2. Case: $z \in D_2$ $(-\pi/4 < \arg k < 0)$ and $\operatorname{Im}\xi < 0$. Using the Airy function Ai, we have

$$\begin{aligned}\mu(R, c; 1) \sim\ & 2^{7/6}3^{1/6}\sqrt{\pi}\, k^{1/6}\xi^{1/6}\left(\frac{z}{z-1}\right)^{1/4} \\ & \times\left[-\frac{\mathrm{e}^{\mathrm{i}\pi k-k}k^k}{\Gamma(1+k)}\mathrm{Ai}((6k)^{2/3}\xi^{2/3}) + \frac{\mathrm{e}^{\mathrm{i}\pi/6+k}k^{-k}}{\Gamma(1-k)}\mathrm{Ai}\big((6k)^{2/3}\xi^{2/3}\mathrm{e}^{2\mathrm{i}\pi/3}\big)\right]\end{aligned} \tag{3.4}$$

in the limit (2.5) (Skovgaard 1966, (5.2)). Since

$$\frac{1}{\Gamma(1+k)} \sim \frac{\mathrm{e}^k}{\sqrt{2\pi}\,k^{k+1/2}}, \quad \frac{1}{\Gamma(1-k)} = \frac{\Gamma(k)\sin\pi k}{\pi} \sim \sqrt{\frac{2}{\pi}}\,\frac{k^{k-1/2}\sin\pi k}{\mathrm{e}^k} \tag{3.5}$$

(Carlson 1977, §§ 3.8 and 3.9), $\sin\pi k \sim -\mathrm{e}^{\mathrm{i}\pi(k+1/2)}/2$ and

$$\mathrm{Ai}(s) + \mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}\big(s\mathrm{e}^{2\mathrm{i}\pi/3}\big) + \mathrm{e}^{-2\mathrm{i}\pi/3}\mathrm{Ai}\big(s\mathrm{e}^{-2\mathrm{i}\pi/3}\big) = 0 \quad \text{for } s \in \mathbb{C}$$

(Abramowitz & Stegun 1964, (10.4.7)), we can rewrite (3.4) in the form

$$\mu(R, c; 1) \sim \frac{2^{2/3}3^{1/6}\xi^{1/6}\mathrm{e}^{\mathrm{i}\pi(k-2/3)}}{k^{1/3}}\left(\frac{z}{z-1}\right)^{1/4}\mathrm{Ai}\big((6k)^{2/3}\xi^{2/3}\mathrm{e}^{-2\mathrm{i}\pi/3}\big) \tag{3.6}$$

in the limit (2.5). Note that (3.6) remains valid even if $z \to 1$ (i.e. $c \to 0$). Indeed,

$$\xi = \tfrac{1}{3}(z-1)^{3/2}[1 + O(z-1)] \tag{3.7}$$

as $z \to 1$ in D_2 (Skovgaard 1966, (3.13)), and thus $\xi^{1/6}$ cancels $(z-1)^{1/4}$ in (3.6).

3. Case: $z \in \ell$ ($\arg k = 0$) and $\operatorname{Im}\xi < 0$. The asymptotic forms (3.4) and (3.5) remain valid, while $\sin \pi k \nsim -\mathrm{e}^{\mathrm{i}\pi(k+1/2)}/2$. Since $|\mathrm{Ai}(s)| \to 0$ as $|s| \to \infty$ with $|\arg s| < \pi/3$ and $|\mathrm{Ai}(s)| \to \infty$ as $|s| \to \infty$ with $\pi/3 < |\arg s| < \pi$ (Abramowitz & Stegun 1964, (10.4.59)), the first (resp. second) term inside the square brackets in (3.4) is dominant over the other if $\arg\xi < -\pi/2$ (resp. $\arg\xi > -\pi/2$). It is not difficult to see that the conditions $\arg\xi \lesseqgtr -\pi/2$ on $\xi(\ell)$ are equivalent to $|z| \lesseqgtr \rho_0$ on ℓ. Therefore,

$$\mu(R, c; 1) \sim \frac{-2^{2/3}3^{1/6}\xi^{1/6}\mathrm{e}^{\mathrm{i}\pi k}}{k^{1/3}} \left(\frac{z}{z-1}\right)^{1/4} \mathrm{Ai}\big((6k)^{2/3}\xi^{2/3}\big) \quad \text{if } |z| < \rho_0, \tag{3.8a}$$

$$\mu(R, c; 1) \sim \frac{2^{5/3}3^{1/6}\xi^{1/6}\mathrm{e}^{\mathrm{i}\pi/6}}{k^{1/3}} (\sin \pi k) \left(\frac{z}{z-1}\right)^{1/4} \mathrm{Ai}\big((6k)^{2/3}\xi^{2/3}\mathrm{e}^{2\mathrm{i}\pi/3}\big) \quad \text{if } |z| > \rho_0, \tag{3.8b}$$

in the limit (2.5). If $|z| = \rho_0$ (i.e. $\xi = \xi_0$), then both terms inside the square brackets in (3.4) have absolute values of the same order, and thus $\mu(R, c; 1)$ is asymptotically equal to the sum of the right sides of (3.8a, b).

4. Case: $z \in D_2 \cup \ell$ and $\operatorname{Im}\xi \geqslant 0$. This case is only for a neighbourhood of $z = \mathrm{e}^{-\mathrm{i}\pi/4}/\sqrt{2}$ (i.e. $c = -\mathrm{i}$), which corresponds to the end point of $\xi(\ell)$ in figure 2(b). The validity of (3.4) in $D_2 \cup \ell$, (3.6) in D_2 and (3.8a) in ℓ is retained if $\xi^{1/6}$ and $\xi^{2/3}$ are replaced by $\xi^{1/6}\mathrm{e}^{-\mathrm{i}\pi/3}$ and $-\xi^{2/3}\mathrm{e}^{-\mathrm{i}\pi/3}$, respectively. However, the case $\operatorname{Im}\xi \geqslant 0$ does not have influence on the results of this paper.

4. Y-shaped structure of c

The zeros of the right sides of (3.3), (3.6) and (3.8b) approximately determine c of (2.4) in the limit (2.5) and, as will be seen next, correspond to mean, wall and centre modes, respectively. The right side of (3.8a) has no zero.

All zeros of I_1 and Ai are located on the imaginary axis and the negative real axis, respectively (Abramowitz & Stegun 1964, §§9.5, 9.6 and 10.4). Taking the regions of k and ξ into account, we deduce that $\arg(k\xi) = -\pi/2$ (which leads to $\arg(k^{2/3}\xi^{2/3}\mathrm{e}^{-2\mathrm{i}\pi/3}) = -\pi$) is necessarily satisfied by all z that make the right side of (3.3) or (3.6) vanish. It leads to

$$\arg\left\{z^{1/2}(z-1)^{1/2} - \ln\left[z^{1/2} + (z-1)^{1/2}\right] - \mathrm{i}\frac{\pi}{2}\right\} - \arg z + \frac{\pi}{4} = 0 \quad \text{for } z \in D_1, \tag{4.1a}$$

$$\arg\left\{z^{1/2}(z-1)^{1/2} - \ln\left[z^{1/2} + (z-1)^{1/2}\right]\right\} - \arg z + \frac{\pi}{4} = 0 \quad \text{for } z \in D_2. \tag{4.1b}$$

Moreover, we add the condition

$$|z| > \rho_0 \quad \text{for } z \in \ell, \tag{4.2}$$

which makes (3.8b) valid. By numerically solving (4.1a, b) and adding the straight line segment given by (4.2), $\arg(1-c) = \pi/4$ with $|1-c| < \rho_0^{-1}$, we obtain the Y-shaped contour shown in figure 3, on which discrete eigenvalues are approximately located. The eigenvalues computed by Schmid & Henningson (1994 (table 1, $n = 0$), 2001 (p. 506, $n = 0$)) are shown in figure 3, too; most are on or near the Y-shaped contour. Although some of their eigenvalues are off the leftward branch of the contour,

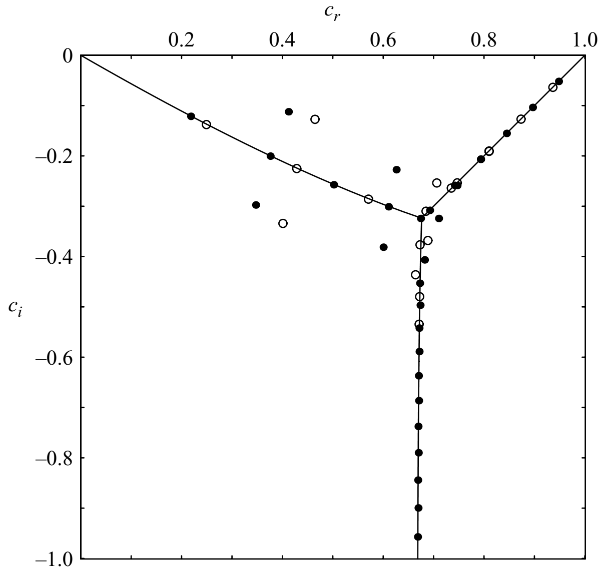

FIGURE 3. The Y-shaped contour obtained from (4.1a, b) and (4.2), with eigenvalues computed by Schmid & Henningson (1994, 2001) for $R = 3000$ (●) and $R = 2\,000$ (○) when $\alpha = 1$.

they are not of torsional disturbances but of meridional disturbances (see Corcos & Sellars 1959, § 9; Burridge & Drazin 1969, (14) and (15)). The downward branch of the contour is given by (4.1a) while the leftward branch by (4.1b). They both look straight at first sight, but in fact neither is straight. The three branches meet at a point $c = c_0$ with

$$c_0 = 1 - \rho_0^{-1} \mathrm{e}^{\mathrm{i}\pi/4} \approx 0.67629 - 0.32371\,\mathrm{i},$$

which was shown in figure 1. Indeed, when $\arg z \to -\pi/4$ in D_1 or D_2, both of (4.1a, b) become equivalent to (3.1) with $\rho_0 = |z|$.

Since (4.1a, b) and (4.2) do not contain α or R but contain only z, the Y-shaped contour in figure 3 is independent of α and R. As will be seen in §§ 5 and 6, the value of αR determines the approximate location of each individual eigenvalue on the contour.

5. Distribution of c for mean modes

The phase velocity $c_r \approx 2/3$ of mean modes was obtained in numerical calculations of the researchers cited in § 1. In the case of c of (2.4), there are several ways to derive $c \to 2/3 - \mathrm{i}\infty$ analytically. One is the way in which the result of Pekeris (1948, (20) and (21)) is used. Second is the method of Burridge & Drazin (1969, (10)–(13)). Let us prove rigorously the following theorem by a third method:

THEOREM 5.1. *Let $R > 0$ be an arbitrarily fixed number, which is not required to be large. Then there exist sequences of c of* (2.4) *such that $c_i \to -\infty$. For all these sequences, $c_r \to 2/3$ holds.*

Proof. The limit $c_i \to -\infty$ with fixed R means (2.5) and $z \to 0$. Noting that $z = 0$ is accessible only in D_1, we first prove the theorem for $c = 1 - 1/z$ with $z \in D_1$ satisfying

$I_1(4k\xi)=0$ i.e. $J_1(4k\xi e^{i\pi/2})=0$ (refer to (3.3)). The error in (3.3) will be discussed in the latter part of this proof.

We denote the absolute value and the argument of the solution z in (4.1a) by ρ and σ, respectively. Let us show that $\sigma \to -\pi/2$ and $d\sigma/d\rho \to 1/3$ as $\rho \to +0$. For $z \in D_1$ with $|z|$ small enough, (3.2) is written as

$$\frac{\xi}{z^{1/2}e^{-i\pi/2}} = \tfrac{1}{2}(1-z)^{1/2} + \tfrac{1}{2}\frac{\ln\left[(-z)^{1/2}+(1-z)^{1/2}\right]}{(-z)^{1/2}} = 1 - \frac{z}{6} + O(z^2). \tag{5.1}$$

It leads to

$$\arg\xi = \tfrac{1}{2}\arg z + \arg\left[1 - \frac{z}{6} + O(z^2)\right] - \frac{\pi}{2}.$$

The definition of k gives $\arg k = -\arg z - \pi/4$. Therefore, from (4.1a), that is $\arg(k\xi) = -\pi/2$ with $z=\rho e^{i\sigma}$, we have

$$\sigma = 2\arg\left[1 - \frac{\rho e^{i\sigma}}{6} + O(\rho^2)\right] - \frac{\pi}{2}. \tag{5.2}$$

Imagine the two surfaces $\tau = 2\arg[1-\rho e^{i\sigma}/6 + O(\rho^2)]$ and $\tau = \sigma + \pi/2$ in the (ρ, σ, τ) space, and think about their intersection. Then, since $\arg[1-\rho e^{i\sigma}/6 + O(\rho^2)] \to +0$ uniformly in $\sigma \in (-\pi/2, -\pi/4)$ as $\rho \to +0$, we see that (5.2) has a solution σ for an arbitrarily small ρ such that $\sigma \to -\pi/2 + 0$ as $\rho \to +0$. Furthermore, applying the implicit function theorem to (5.2) and noting that $\arg \equiv \operatorname{Im}\ln$, we have

$$\lim_{\rho\to+0}\frac{d\sigma}{d\rho} = \lim_{\rho\to+0}\frac{2\operatorname{Im}(\partial/\partial\rho)\ln(1-\rho e^{i\sigma}/6)}{1-2\operatorname{Im}(\partial/\partial\sigma)\ln(1-\rho e^{i\sigma}/6)} = \lim_{\sigma\to-\pi/2} 2\operatorname{Im}\left(-\frac{e^{i\sigma}}{6}\right) = \tfrac{1}{3}.$$

From the limits $\sigma \to -\pi/2$ and $d\sigma/d\rho \to 1/3$, we can verify that $\rho e^{i\sigma}$ with ρ small enough is really located in D_1, that is in the region between the arc of $|z-1/2|=1/2$ (or equivalently $|z| = \cos\arg z$) and the negative imaginary axis on the z-plane. Those limits yield

$$1 - \frac{1}{\rho e^{i\sigma}} = 1 - \frac{\cos\sigma}{\rho} + i\frac{\sin\sigma}{\rho} \to \tfrac{2}{3} - i\infty \quad \text{as } \rho \to +0. \tag{5.3}$$

We have $\xi = O(z^{1/2})$ in D_1 as $z \to 0$ (refer to (5.1)), while $k = O(z^{-1})$. This means that $|k\xi| \to \infty$ as $z \to 0$. Hence we can define sequences $\{\rho_n : \rho_n > 0, \lim_{n'\to\infty}\rho_{n'} = 0\}$ and $\{\sigma_n\}$ with $n \in \mathbb{N}$ large enough such that $z = \rho_n e^{i\sigma_n} \in \{\rho e^{i\sigma}\}$ satisfies $4k\xi = j_{1,n}e^{-i\pi/2}$, for which $I_1(4k\xi)=0$. Here $j_{1,n}$ denotes the nth zero of the Bessel function J_1. It follows from (5.3) that $1-(\rho_n e^{i\sigma_n})^{-1} \to 2/3 - i\infty$ as $n \to \infty$. Except this sequence and its subsequences, there exists no sequence of c such that $c_i \to -\infty$.

As we saw, the terms $1 - z/6$ in (5.1) are crucial for deriving the limit $c \to 2/3 - i\infty$ from $I_1(4k\xi)=0$. In fact, the same limit is deduced even from

$$I_1\left(4kz^{1/2}e^{-i\pi/2}\left[1 - \left(\tfrac{1}{6} + i\beta\right)z + o(z)\right]\right) = 0 \tag{5.4}$$

with any fixed $\beta \in \mathbb{R}$. According to Skovgaard (1966, (5.6) and (4.26) with (4.14), (4.18) and (4.36)), a more detailed form of (3.3) is

$$\mu(R, c; 1) = M_{\kappa,1/2}(4\kappa z_*) = 2(-\xi_*)^{1/2}\left(\frac{z_*}{z_*-1}\right)^{1/4}\Sigma$$

with

$$\Sigma = \left(1 + \frac{A_2}{|\kappa|^2} + \frac{A_4}{|\kappa|^4} + \cdots\right) I_1(4\kappa\xi_*) + \frac{1}{|\kappa|}\left(B_0 + \frac{B_2}{|\kappa|^2} + \frac{B_4}{|\kappa|^4} + \cdots\right) I_2(4\kappa\xi_*).$$

n	$1-(\rho_n e^{i\sigma_n})^{-1}$	c_{SH}
12	0.6731–0.4526 i	0.672883–0.453187 i
13	0.6717–0.5416 i	0.671515–0.542212 i
14	0.6706–0.6364 i	0.670474–0.636976 i
15	0.6698–0.7371 i	0.669683–0.737636 i
16	0.6692–0.8438 i	0.669078–0.844317 i
17	0.6687–0.9567 i	0.668611–0.957120 i

TABLE 1. Several values of $1-(\rho_n e^{i\sigma_n})^{-1}$ for $\alpha R = 3000$ and the corresponding eigenvalues c_{SH} of Schmid & Henningson (1994).

Here $z_* = kz/\kappa = z/(1 - i\alpha z/R) \in D_1$, $\xi_* = \xi(z_*)$, and $A_2, A_4, \ldots;\ B_0, B_2, B_4 \ldots$ are functions of z_* and $e^{i\arg\kappa}$ such that, for $\lambda \in \mathbb{N}$, $A_{4\lambda-2} = O(\xi_*^2)$, $A_{4\lambda} = O(\xi_*^4)$, $B_{4(\lambda-1)} = O(\xi_*^3)$ and $B_{4\lambda-2} = O(\xi_*)$ as $\xi_* \to 0$. Noting $(d/ds)I_1(s) = I_2(s) + I_1(s)/s$ and $(d/ds)I_2(s) = I_1(s) - 2I_2(s)/s$, we can write

$$\Sigma = \left(1 + \frac{a_2}{|\kappa|^2} + \frac{a_4}{|\kappa|^4} + \cdots\right) I_1\left(4\kappa\xi_* + \frac{1}{|\kappa|}\left(B_0 + \frac{b_2}{|\kappa|^2} + \frac{b_4}{|\kappa|^4} + \cdots\right)\right),$$

where $a_2, a_4, \ldots; b_2, b_4, \ldots$ are determined with $A_2, A_4, \ldots;\ B_0, B_2, B_4 \ldots$; and $e^{i\arg\kappa}\xi_*$. Since

$$\kappa\xi_* = \kappa z_*^{1/2} e^{-i\pi/2}\left[1 - \frac{z_*}{6} + O(z_*^2)\right] = kz^{1/2} e^{-i\pi/2}\left[1 - \frac{z}{6} - \frac{i\alpha z}{2R} + O(z^2)\right],$$

$|\kappa|^{-1} = O(k^{-1}) = O(z)$ and $B_0 = O(z_*^{3/2}) = O(z^{3/2})$, an equality in the form (5.4) follows from $\Sigma = 0$. Therefore, the error in (3.3) does not have influence on the limit $c \to 2/3 - i\infty$. The proof is complete. □

Henceforth, to the end of § 6, R is assumed to be so large that (3.3), (3.6) and (3.8b) have negligible errors except in the neighbourhoods of the boundaries of D_1, D_2 and ℓ, respectively, while $|c_i|$ is not required to be large.

We continue to use the above-defined ρ, σ, ρ_n and σ_n; that is $\rho e^{i\sigma} \in D_1$ is a solution of (4.1a); $\{\rho_n e^{i\sigma_n}\} \subset \{\rho e^{i\sigma}\}$; and

$$\frac{|\xi(\rho_n e^{i\sigma_n})|}{\rho_n} = \frac{j_{1,n}}{\sqrt{\alpha R}}. \tag{5.5}$$

Table 1 shows several values of $1-(\rho_n e^{i\sigma_n})^{-1}$ for $\alpha R = 3000$ and the corresponding eigenvalues c_{SH} computed by Schmid & Henningson (1994, table 1, $n = 0$). Their agreement in real and imaginary parts is so good that the differences are of order 10^{-4}. It is expected that $1-(\rho_n e^{i\sigma_n})^{-1}$ is closer to the true value of c for larger αR or n.

Let us see how each c moves nearly on the downward branch of the Y-shaped contour in figure 3 when R becomes still larger. For this, note that c is determined from n and αR through the following process:

(i) The integer n is fixed so large that

$$\frac{j_{1,n}}{\sqrt{\alpha R}} \gtrsim \frac{|\xi_0 - i\pi/4|}{\rho_0} \approx 0.61228 \tag{5.6}$$

(see § 8.2). For example $n \geqslant 11$ for $\alpha R = 3000$, and $n \geqslant 9$ for $\alpha R = 2000$.

(ii) The value of $|\xi(\rho_n e^{i\sigma_n})|/\rho_n$ is determined by (5.5).

m	$1-(\widehat{\rho}_m e^{i\widehat{\sigma}_m})^{-1}$	c_{SH}
1	0.2185–0.1207 i	0.21843581–0.121310028 i
2	0.3763–0.1998 i	0.3762424–0.2004630 i
3	0.5021–0.2567 i	0.502037–0.257316 i
4	0.6111–0.3004 i	0.61086–0.301052 i

TABLE 2. Several values of $1-(\widehat{\rho}_m e^{i\widehat{\sigma}_m})^{-1}$ for $\alpha R = 3000$ and the corresponding eigenvalues c_{SH} of Schmid & Henningson (1994).

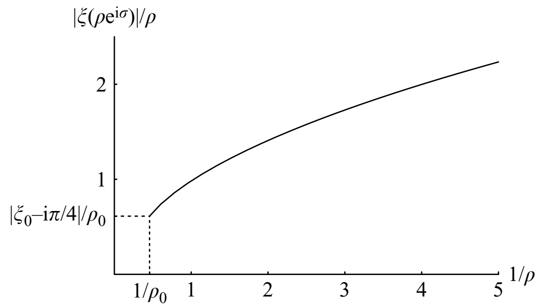

FIGURE 4. The relation between $1/\rho$ and $|\xi(\rho e^{i\sigma})|/\rho$.

(iii) The value of $1/\rho_n$ is given by figure 4, which shows the relation between $1/\rho$ and $|\xi(\rho e^{i\sigma})|/\rho$.

(iv) The approximate location of c on the downward branch of the Y-shaped contour in figure 3 is obtained from $|c-1| \sim 1/\rho_n$.

Therefore, if R becomes larger with n fixed, then $1/\rho_n$ becomes smaller, and the location of c goes up towards the bifurcation point c_0.

6. Distribution of c for wall and centre modes

6.1. *Wall modes*

Let $\widehat{\rho}$ and $\widehat{\sigma}$ be the absolute value and the argument of the solution $z \in D_2$ in (4.1*b*), that is $\arg(k\xi) = -\pi/2$. In addition, denoting the mth negative zero of Ai by a_m (for its value, see Abramowitz & Stegun 1964, table 10.13), we define $\widehat{\rho}_m$ and $\widehat{\sigma}_m$ by $\{\widehat{\rho}_m e^{i\widehat{\sigma}_m}\} \subset \{\widehat{\rho} e^{i\widehat{\sigma}}\}$ and

$$\frac{|\xi(\widehat{\rho}_m e^{i\widehat{\sigma}_m})|}{\widehat{\rho}_m} = \frac{2|a_m|^{3/2}}{3\sqrt{\alpha R}}. \tag{6.1}$$

Then $z = \widehat{\rho}_m e^{i\widehat{\sigma}_m}$ satisfies $6k\xi = |a_m|^{3/2} e^{-i\pi/2}$, for which the right side of (3.6) vanishes. Table 2 shows several values of $1-(\widehat{\rho}_m e^{i\widehat{\sigma}_m})^{-1}$ for $\alpha R = 3000$ and the corresponding eigenvalues c_{SH} computed by Schmid & Henningson (1994, table 1, $n=0$).

The location of c for wall modes is obtained from m and αR in the following way:

(i) The integer $m \geqslant 1$ is fixed so that

$$\frac{2|a_m|^{3/2}}{3\sqrt{\alpha R}} \lessapprox \frac{|\xi_0|}{\rho_0} \approx 0.25272 \tag{6.2}$$

is satisfied. For example $1 \leqslant m \leqslant 4$ for $\alpha R = 3000$ and $1 \leqslant m \leqslant 3$ for $\alpha R = 2000$. The largest m for given αR means the number of wall modes.

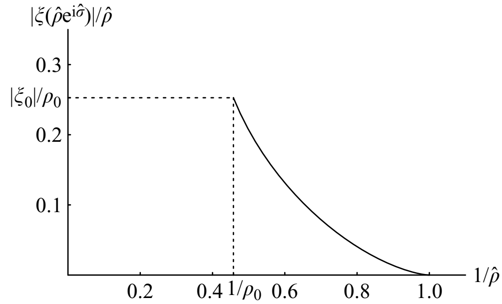

FIGURE 5. The relation between $1/\widehat{\rho}$ and $|\xi(\widehat{\rho}e^{i\widehat{\sigma}})|/\widehat{\rho}$.

(ii) The value of $|\xi(\widehat{\rho}_m e^{i\widehat{\sigma}_m})|/\widehat{\rho}_m$ is determined by (6.1).

(iii) The value of $1/\widehat{\rho}_m$ is given by figure 5, which shows the relation between $1/\widehat{\rho}$ and $|\xi(\widehat{\rho}e^{i\widehat{\sigma}})|/\widehat{\rho}$.

(iv) The approximate location of c on the leftward branch of the Y-shaped contour in figure 3 is obtained from $|c-1| \sim 1/\widehat{\rho}_m$.

Therefore, if R becomes larger with m fixed, then $1/\widehat{\rho}_m$ becomes larger, and the location of c moves towards the origin.

Particularly, in the case in which c is close to the origin, that is z is close to unity, (3.7) with $k \sim \sqrt{\alpha R}\, e^{-i\pi/4}/4$ yields

$$|a_m|^{3/2} e^{-i\pi/2} = 6k\xi|_{z=\widehat{\rho}_m \exp(i\widehat{\sigma}_m)} \sim \frac{\sqrt{\alpha R}\, e^{-i\pi/4}}{2} \left(\widehat{\rho}_m e^{i\widehat{\sigma}_m} - 1\right)^{3/2}.$$

It leads to

$$c \sim 1 - \left(\widehat{\rho}_m e^{i\widehat{\sigma}_m}\right)^{-1} \sim \widehat{\rho}_m e^{i\widehat{\sigma}_m} - 1 \sim 2^{2/3}|a_m|(\alpha R)^{-1/3} e^{-i\pi/6}.$$

This is an already-known fact about wall modes (Burridge & Drazin 1969, in which the minus sign of z_q in either (14) or (15) should be removed).

6.2. *Centre modes*

Because of the factor $\sin \pi k$ in (3.8*b*), the distribution of c for centre modes is approximately given by $k \in \mathbb{N}$ if k is large. If k is not large, that is c close to unity, we need another asymptotic form valid for $z \in D_1 \cup D_2 \cup \ell$:

$$\mu(R, c; 1) \sim \frac{\Gamma(k) \sin \pi k}{\pi} (4kz)^{-k} e^{2kz} \quad \text{as } R \to \infty \text{ with } \sqrt{R}\,(1-c) \text{ fixed.} \tag{6.3}$$

This follows from an asymptotic expansion of a confluent hypergeometric function for a large-modulus variable and fixed parameters (Abramowitz & Stegun 1964, (13.5.1); Buchholz 1969, §7.1, (3)), which Pekeris (1948, (42)) also used. The right side of (6.3) also vanishes for $k \in \mathbb{N}$. Consequently,

$$c \sim 1 - \frac{4\nu e^{i\pi/4}}{\sqrt{\alpha R}} \quad \text{for every } \nu \in \mathbb{N} \text{ that satisfies} \quad \nu \lessapprox \frac{\sqrt{\alpha R}}{4\rho_0} \tag{6.4}$$

is obtained from $k \sim \nu$. This form of c is well known, but ν has mostly been written as $\nu = 1, 2, 3, \ldots$ without an upper bound. The analytical deduction of the upper bound $\sqrt{\alpha R}/(4\rho_0)$ from (4.2) is new. For example $1 \leqslant \nu \leqslant 6$ for $\alpha R = 3000$, and $1 \leqslant \nu \leqslant 5$ for $\alpha R = 2000$. It is easy to verify the good agreement of (6.4) with eigenvalues computed by Schmid & Henningson (1994 (table 1, $n=0$), 2001 (p. 506, $n=0$)). Clearly, the approximate location of c moves towards unity on the rightward straight branch in figure 3 when R becomes larger with ν fixed.

7. Mean modes for meridional disturbances

For axisymmetric meridional disturbances, complex phase velocities c are given by

$$\int_0^1 \mu(R, c; r) I_1(\alpha r)\,\mathrm{d}r = 0 \tag{7.1}$$

(Pekeris 1948, (17)). They are more complicated than c given by (2.4). Nevertheless, it is not difficult to investigate analytically their distribution for mean modes.

THEOREM 7.1. *Theorem 5.1 is also true with c of* (7.1).

Let us give an outline of the proof of this theorem. In the same way as (3.3), it follows from a result of Skovgaard (1966) that

$$\mu(R, c; r) \sim M_{k,1/2}(4kzr^2) \sim 2(-\widetilde{\xi})^{1/2}\left(\frac{zr^2}{zr^2-1}\right)^{1/4} I_1(4k\widetilde{\xi}) \qquad \text{with } \widetilde{\xi} = \xi(zr^2)$$

for $z \in D_1$ and $r \in (0, 1)$ in the limit (2.5), where the definition of ξ on D_1 is extensionally used for $zr^2 \notin D_1$. This form of $\mu(R, c; r)$ is the same as the one Reid & Ng (2003, (20)) obtained, except a constant factor. Since $\mathrm{d}\widetilde{\xi}/\mathrm{d}r = z^{1/2}(zr^2-1)^{1/2}$,

$$I_1(4k\widetilde{\xi}) = \frac{(\mathrm{d}/\mathrm{d}r)I_0(4k\widetilde{\xi})}{4kz^{1/2}(zr^2-1)^{1/2}}, \qquad I_0(4k\widetilde{\xi}) = \frac{(\mathrm{d}/\mathrm{d}r)I_1(4k\widetilde{\xi})}{4kz^{1/2}(zr^2-1)^{1/2}} + \frac{I_1(4k\widetilde{\xi})}{4k\widetilde{\xi}}, \tag{7.2}$$

integration by parts yields

$$\begin{aligned}
&\int_0^1 \mu(R, c; r) I_1(\alpha r)\,\mathrm{d}r \\
&\quad\sim \frac{(-\xi)^{1/2} I_0(4k\xi) I_1(\alpha)}{2kz^{1/4}(z-1)^{3/4}} - \frac{1}{2kz^{1/4}}\int_0^1 I_0(4k\widetilde{\xi})\frac{\mathrm{d}}{\mathrm{d}r}\frac{(-\widetilde{\xi}\,r)^{1/2} I_1(\alpha r)}{(zr^2-1)^{3/4}}\,\mathrm{d}r \\
&\quad= \frac{(-\xi)^{1/2} I_0(4k\xi) I_1(\alpha)}{2kz^{1/4}(z-1)^{3/4}} - \frac{I_1(4k\xi)}{8k^2z^{3/4}(z-1)^{1/2}}\left.\frac{\mathrm{d}}{\mathrm{d}r}\frac{(-\widetilde{\xi}\,r)^{1/2} I_1(\alpha r)}{(zr^2-1)^{3/4}}\right|_{r=1} \\
&\qquad+ \frac{1}{8k^2z^{3/4}}\int_0^1 I_1(4k\widetilde{\xi})\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{(zr^2-1)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}r}\frac{(-\widetilde{\xi}\,r)^{1/2} I_1(\alpha r)}{(zr^2-1)^{3/4}}\right]\mathrm{d}r \\
&\qquad- \frac{1}{8k^2z^{1/4}}\int_0^1 \frac{I_1(4k\widetilde{\xi})}{\widetilde{\xi}}\frac{\mathrm{d}}{\mathrm{d}r}\frac{(-\widetilde{\xi}\,r)^{1/2} I_1(\alpha r)}{(zr^2-1)^{3/4}}\,\mathrm{d}r
\end{aligned} \tag{7.3}$$

for $z \in D_1$ in the limit (2.5). As $|k| \to \infty$ with $kz = \sqrt{\alpha R}\,\mathrm{e}^{-\mathrm{i}\pi/4}/4$ fixed, we can show that (7.3) is equal to

$$-\frac{I_1(\alpha)}{2k}\left[I_0(4k\xi) - \frac{\mathrm{i}\alpha}{4kz^{1/2}}\frac{I_0(\alpha)}{I_1(\alpha)}I_1(4k\xi) + O(k^{-1})I_0(4k\xi) + O\left(k^{-3/2}\right)I_1(4k\xi)\right]$$

by using $(-\widetilde{\xi})^{1/2} = z^{1/4}\mathrm{e}^{\mathrm{i}\pi/4}r^{1/2}[1 + O(k^{-1})]$ and $(zr^2-1)^{1/4} = \mathrm{e}^{-\mathrm{i}\pi/4}[1 + O(k^{-1})]$ and (7.2) again. The factor $-\alpha/(4kz^{1/2})$ equals $4kz^{1/2}\mathrm{e}^{-\mathrm{i}\pi/2}z/R$. Therefore, (7.1) leads to an equality in the form of (5.4) with I_1 replaced by I_0, and thus the limit $c \to 2/3 - \mathrm{i}\infty$ is deduced in the same way as in the proof of theorem 5.1 by replacing $j_{1,n}$ by $j_{0,n}$ (the nth zero of J_0).

n	$1-1/z_n$	c_{SH}
13	0.6723–0.4968 i	0.6736237–0.496799 i
14	0.6711–0.5887 i	0.6717237–0.588886 i
15	0.6702–0.6864 i	0.6708160–0.6864667 i
16	0.6695–0.7901 i	0.67015960–0.7899999 i
17	0.6689–0.8998 i	0.6696446–0.8996122 i

TABLE 3. Several values of $1-1/z_n$ for $\alpha R=3000$ and the corresponding eigenvalues c_{SH} of Schmid & Henningson (1994).

When R is large instead of $|c_i|$, that is $|k|\to\infty$ with $|z|$ and $|z|^{-1}$ bounded, it also follows from (7.3) that approximate values of c of (7.1) for mean modes are given by $I_0(4k\xi)=0$. Denoting the solution of $4k\xi=j_{0,n}\mathrm{e}^{-\mathrm{i}\pi/2}$ by z_n, we obtain table 3 for $\alpha R=3000$. It shows good agreement with $1-1/z_n$ and the corresponding eigenvalues c_{SH} of Schmid & Henningson (1994, table 1, $n=0$). The well-known interlacing of $j_{1,n}$ and $j_{0,n}$ accounts for the alternate distribution of c for torsional and meridional disturbances, which was numerically observed by Salwen & Grosch (1972) and O'Sullivan & Breuer (1994).

8. Concluding remarks

8.1. *On asymptotic forms of* $M_{k,1/2}(s)$

It is obvious that (3.3), (3.6) and (3.8*b*) have played essential roles in this paper. They follow from asymptotic forms of $M_{k,1/2}(s)$ for $|k|\to\infty$ with $s/k\in\mathbb{C}$ finite, which are part of the results of Skovgaard (1966). Pekeris (1948, (42)) used an asymptotic form for $|s|\to\infty$ with k fixed, and we also used it in (6.3). Sexl & Spielberg (1958, (46)) applied an asymptotic form for $|k|\to\infty$ with s fixed (Buchholz 1969, § 7.4, (16)) to meridional disturbances. In the torsional case, it reads

$$\mu(R,c;1)\sim\frac{2}{(1-c)^{1/2}}J_1\left(\sqrt{\alpha R}\,(1-c)^{1/2}\mathrm{e}^{-\mathrm{i}\pi/4}\right)\quad\text{as } |1-c|\to\infty \text{ with } R \text{ fixed.}\qquad(8.1)$$

From this, we cannot deduce the limit $c_r\to 2/3$ in theorem 5.1 but only the limit $\arg(1-c)\to\pi/2$ as $|1-c|\to\infty$. In the proof of theorem 5.1, the term $-z/6$ in (5.1) is crucial for obtaining $c_r\to 2/3$. Taking this into account, we notice that (8.1) has lost the information of $c_r\to 2/3$. Indeed, (8.1) is also derived from (3.3) with $\xi\sim z^{1/2}\mathrm{e}^{-\mathrm{i}\pi/2}$, which follows from (5.1) with $O(z)$ neglected.

8.2. *On the sign* $\nsim$ *in* (5.6), (6.2) *and* (6.4)

The change of the approximate location of c on each branch of the Y-shaped contour in figure 3 has been explained in §§ 5 and 6. However, the transition of c from the mean-mode (downward) branch to the centre-mode (rightward) or the wall-mode (leftward) branch has not been made clear. The reason is that the errors in (3.3), (3.6) and (3.8*b*) are not negligible in a neighbourhood (which vanishes as $R\to\infty$) of the bifurcation point c_0. The sign $\nsim$ in (5.6), (6.2) and (6.4) implies this fact. Indeed, $j_{1,11}/\sqrt{3000}\approx 0.64508$ is larger than the right side of (5.6), but $1-(\rho_{11}\mathrm{e}^{\mathrm{i}\sigma_{11}})^{-1}$ does not agree with any of c_{SH} unlike $1-(\rho_n\mathrm{e}^{\mathrm{i}\sigma_n})^{-1}$ for $n\geqslant 12$ in table 1.

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